The Role of Connections as Minimal Norms in Normative Systems

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Abstract. The paper aims at developing an algebraic representation of normative systems by sets of minimal norms, "connections", within relational structures called condition implication structures. It is shown that, given some general presuppositions, a normative system is completely determined by its set of connections, and that comparisons between normative systems can be made by considering whether connections in one system are narrower or wider than in another. The general framework for the study is Boolean algebra and a relational structure called Boolean quasi-ordering.

1 Introduction

In 1971, Alchourrón and Bulygin published their important book *Normative Systems* [1], which contains a logical and model-theoretical analysis of systems of norms. Partly influenced by that book, the present authors have developed an algebraic theory of normative systems based on Boolean quasi-orderings.

A few examples of what can be regarded as desiderata for a norm representation method are as follows.

1. The system of norms is depicted in a lucid, concise and effective way.
2. Changes and extensions of the normative system are easily described.
3. The normative system can be divided in different parts which can be changed independently.

In a number of papers ([6, 7, 8, 9, 12]) by the present authors, an algebraic representation of normative systems has been developed, namely in a theory of *condition implication structures* (*cis*'s) using methods from algebra and set theory. In this approach, normative systems are studied essentially as deductive mechanisms yielding outputs for inputs. Insofar, our representation is similar to that in [1]. A difference, however, is that, in our approach, while input and output are particular, norms are explicitly general in character. (See the comments on this in [10].) For another approach to the same problem area, see Makinson and van der Torre [11, 12].

A logically satisfactory theory of implication between conditions requires a general logical theory as its basis. In our work, this basis is presented in the form of a theory of *Boolean quasi-orderings*, or *Bqo*'s for short. The *Bqo* theory is a general algebraic theory having models of many kinds. As will appear, the theory of *cis*'s is one of its models.

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The deontic fine structure of normative conditions can be expressed by means of the so-called Kanger-Lindahl theory of normative positions [5]. This theory, based on deontic logic and the logic of action, has recently been further developed by Marek Sergot and Andrew J.I. Jones, see [2, 3, 13, 14]. In [9], the present authors took a step towards integrating the theory of normative positions in the *cis* or *Bqo* representation of normative systems.

In the present paper, while building upon the assumption that descriptive and normative conditions are two different kinds, we will not deal with the fine structure of normative conditions. (We refer the reader to [9] as regards this topic.) Our aim is different, namely to exhibit some features of our representation that makes it meet the desiderata mentioned above. Thus a central notion in our theory is that of *connection*, in the sense of a set of particularly qualified implications from a purely descriptive to a purely normative part of a normative system. Connections can be characterized as a kind of minimal norms in a system, constituting a "minimal codex", which determines the system and from which the other norms can be derived. Due to this result, the notion of a connection is central when comparing two normative systems and central also for the topic of change of a system into a different one.

We shall first discuss the representation of norms by ordered pairs consisting of a ground and a consequence, then briefly sketch an algebraic approach to normative systems. After that, in section 3, we will show the crucial role of connections as minimal norms. Proofs of the two main theorems presented are given in the Appendix. (Proofs of lemmas are omitted.)

2 The Representation of Norms and Normative Systems

2.1 The Representation of Norms

2.1.1 Implicative Relation and Ordered Pairs

In predicate logic, a norm-sentence is (usually) expressed as a universal sentence. For example:

(n_1) For any x, y and z : if x has promised to pay \$$y$ to z then x has an obligation to pay \$$y$ to z.

We can formalize (n_1) within predicate logic as follows, where 'Obligatory' is a deontic operator resulting in a new predicate when it is applied to a given predicate.

(n_2) $\forall x, y, z : \text{Promise}(x, y, z) \longrightarrow \text{ObligatoryPay}(x, y, z)$

Thus, a typical norm-sentence is a universal implication. Syntactically it consists of three parts: the sequence of universal quantifiers, the antecedent formula and the consequent formula. Note that norm (n_2) correlates open sentences:

$$\text{Promise } (x, y, z) \text{ is correlated to ObligatoryPay } (x, y, z).$$

The idea we have elaborated in earlier papers is to represent a norm as a relational statement correlating a ground to a consequence. For example, the norm (n_2) is expressed as

$$\text{Promise } \mathcal{R} \text{ ObligatoryPay}.$$

Generally, $p\mathcal{R}q$ represents the norm

(n_3) $\forall x_1, ..., x_\nu : p(x_1, ..., x_\nu) \longrightarrow q(x_1, ..., x_\nu)$

given that p and q are ν-ary predicates. It is important here that the free variables in $p(x_1, ..., x_\nu)$ are the same and in the same order as the free variables in $q(x_1, ..., x_\nu)$. (However, see [8], p. 263.) $\mathcal{R}$ is a binary relation, and $p\mathcal{R}q$ is a relational statement equivalent to $\langle p, q\rangle \in \mathcal{R}$. Thus, we represent a norm as $p\mathcal{R}q$ or $\langle p, q\rangle \in \mathcal{R}$. Note that the implicative relation $\mathcal{R}$ can be such that only some of the elements are norms. $p\mathcal{R}q$ as a representation of (n_3) does not generally presuppose that q is a normative (or deontic) predicate, so we can use $p\mathcal{R}q$ as a representation of any sentence which has the same form as (n_3).

In the discussion above of the representation of norms, we have spoken about Promise and ObligatoryPay as well as p and q as predicates. But the term predicate is often used for syntactical entities, and, therefore, interpreting $p\mathcal{R}q$, we have chosen to refer to p and q as *conditions*. Thus, grounds and consequences in a norm are represented as conditions, and a norm-sentence is represented as a sentence $p\mathcal{R}q$ (or $\langle p, q\rangle \in \mathcal{R}$) relating, or "correlating", a ground to a consequence. Grounds are descriptive and consequences are normative conditions. If, in a context, it is presupposed that $p\mathcal{R}q$ where p is descriptive and q is normative, we often refer to the ordered pair $\langle p, q\rangle$ as a norm.

2.1.2 A First Remark on Minimal Norms

One norm can follow from another norm. For example, a norm n_2 follows from a norm n_1, which will be written $n_1 \trianglelefteq n_2$, if the ground of n_2 implies the ground of n_1 and the consequence of n_1 implies the consequence of n_2. If $n_1 = \langle p_1, q_1\rangle$ and $n_2 = \langle p_2, q_2\rangle$ then

$$n_1 \trianglelefteq n_2 \text{ if and only if } p_2 \text{ implies } p_1 \text{ and } q_1 \text{ implies } q_2.$$

If we express implications between grounds respectively between consequences in terms of $\mathcal{R}$, we can define $\trianglelefteq$ in the following way:

$$\langle p_1, q_1\rangle \trianglelefteq \langle p_2, q_2\rangle \text{ if and only if } p_2\mathcal{R}p_1 \text{ and } q_1\mathcal{R}q_2.$$

It is easy to see that $\trianglelefteq$ is a quasi-ordering, i.e. transitive and reflexive. Given a normative system we can determine the set of minimal norms for that system (with respect to $\trianglelefteq$), i.e. the set of those norms that do not follow from any other norms. The norms which are minimal with respect to $\trianglelefteq$ are of special interest. These norms will be called connections, and they can be regarded as containing the normative content of the system of norms. Therefore, given some general requirements, a normative system is completely described by its set of minimal norms. This idea is made more precise in what follows.

2.2 Representation of Normative Systems

2.2.1 Introductory Remarks

A normative system is a structure based on a set of norms. To describe the structure we need some preliminary notions. As is easy to see, we can form conjunctions, disjunctions and negations of conditions by $\wedge, \vee, '$ in the following way (where $x_1, ..., x_\nu$ are place-holders, not individual constants).

$(p \wedge q)(x_1, ..., x_\nu)$	if and only if	$p(x_1, ..., x_\nu)$ and $q(x_1, ..., x_\nu)$.
$(p \vee q)(x_1, ..., x_\nu)$	if and only if	$p(x_1, ..., x_\nu)$ or $q(x_1, ..., x_\nu)$.
$(p')(x_1, ..., x_\nu)$	if and only if not	$p(x_1, ..., x_\nu)$.

As is well-known, the truth-functional connectives can be used as operations in Boolean algebras. It is therefore possible to construct Boolean algebras of conditions.

Suppose that we have a set of norms $\{\langle p_i, q_i\rangle : 1 \leq i \leq k\}$. We form three Boolean algebras, viz.

1. the Boolean algebra $\mathcal{B}_1$ of grounds generated by $\{p_1, ..., p_k\}$
2. the Boolean algebra $\mathcal{B}_2$ of consequences generated by $\{q_1, ..., q_k\}$
3. the Boolean algebra $\mathcal{B}_0$ generated by $\{p_1, ..., p_k, q_1, ..., q_k\}$.

$\mathcal{B}_1$ and $\mathcal{B}_2$ are subalgebras of $\mathcal{B}_0$. The set of grounds and the set of consequences are disjoint, i.e. they contain no common element. The norms are links or joinings (within the Boolean algebra $\mathcal{B}_0$) from the Boolean algebra of grounds to the Boolean algebra of consequences. From a formal point of view, the role of the set of norms is thus to join two Boolean subalgebras. As a preliminary result of the discussion in this subsection we can say that the normative system based on a set of norms is a Boolean algebra of conditions and the norms are links or joinings from the subalgebra of grounds to the subalgebra of consequences. However, the structure over the consequences and over the grounds, taken separately as well as together, is more complicated than just a Boolean algebra.

2.2.2 Boolean Quasi-orderings and Condition Implication Structures

To transform the sketch in the preceding subsection into a workable theory, we need a formal framework with great expressive power. Boolean algebra supplies us with such a framework. In our earlier papers we have developed an algebraic approach to the representation of norms. One of the main tools for our analysis of normative systems is the theory of Boolean quasi-orderings.

With the notion of a Boolean algebra as a starting point, it is possible to define a Boolean quasi-ordering as a Boolean algebra extended with a quasi-ordering satisfying certain conditions.

Definition 1. *The relational structure $\langle B, \wedge, ', R\rangle$ is a* Boolean quasi-ordering *if $\langle B, \wedge, '\rangle$ is a Boolean algebra and R is a binary, reflexive and transitive relation on B such that R satisfies the following conditions for all a,b and c in B:*

1. *aRb and aRc implies $aR(b \wedge c)$.*
2. *aRb implies $b'Ra'$.*
3. *$(a \wedge b)Ra$.*
4. *not $\top R \bot$.*

The indifference part of R is denoted Q and is defined by: aQb if and only if aRb and bRa. Similarly, the strict part of R is denoted S and is defined by: aSb if and only if aRb and not bRa.

Let $\leq$ be the partial ordering determined by the Boolean algebra $\langle B, \wedge, '\rangle$. From requirement (3) for Boolean quasi-orderings it follows that $a \leq b$ implies aRb. If $\langle B, \wedge, ', R\rangle$ is a Boolean quasi-ordering then we say that the Boolean algebra $\langle B, \wedge, '\rangle$ is the *reduct* of $\langle B, \wedge, ', R\rangle$, denoted $\mathcal{B}^{red}$.

An important class of models of the theory of Boolean quasi-orderings consists of models having a set of conditions as its domain.

Definition 2. *A* condition implication structure *is a Boolean quasi-ordering $\mathcal{B} = \langle B, \wedge, ', R\rangle$ such that B is a domain of conditions, and R is such that aRb represents that a implies b.*

If a and b are ν-ary conditions, aRb is the representation of

$$\forall x_1, .., x_\nu : a(x_1, ..., x_\nu) \to b(x_1, ..., x_\nu).$$

If $\mathcal{S}$ is a normative system represented by a condition implication structure $\mathcal{B}$, a normative correlation in $\mathcal{S}$ is represented by $a_1 R a_2$, where $a_1, a_2 \in B$, and a_1 is descriptive while a_2 is normative and we say that $\langle a_1, a_2 \rangle$ is a norm in $\mathcal{S}$.

3 The Role of Connections

3.1 Definitions

In the study of normative systems, it is necessary to distinguish between various parts of a Boolean quasi-ordering (corresponding to the set of grounds and the set of consequences), and therefore, an important notion is that of a fragment of a Boolean quasi-ordering.

Definition 3. *If $\mathcal{B} = \langle B, \wedge, ', R \rangle$ is a Boolean quasi-ordering, and $\langle B_i, \wedge, ' \rangle$ is a subalgebra of $\langle B, \wedge, ' \rangle$, and $R_i = R/B_i$, then the structure $\mathcal{B}_i = \langle B_i, \wedge, ', R_i \rangle$ will be called a* fragment *of $\mathcal{B}$.*

Note that if $\mathcal{B}$ is a Boolean quasi-ordering and $\mathcal{B}_i$ is a fragment of $\mathcal{B}$, then $\mathcal{B}_i$ is a Boolean quasi-ordering.

Two kinds of combinations of fragments are “joinings” and “connections”.

Definition 4. *Let $\mathcal{B}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ be Boolean quasi-orderings such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are fragments of $\mathcal{B}$. A* joining *from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ is a pair $\langle b_1, b_2 \rangle$ in $\mathcal{B}$ such that $b_1 \in B_1$, $b_2 \in B_2$, $b_1 R b_2$, not $b_1 R \bot$ and not $\top R b_2$. A joining $\langle b_1, b_2 \rangle$ from $\mathcal{B}_1$ to $\mathcal{B}_2$ is called* strict *if $b_1 S b_2$.*

The set of joinings from $\mathcal{B}_1$ to $\mathcal{B}_2$ (in $\mathcal{B}$) will be denoted $Joining_{\mathcal{B}}(\mathcal{B}_1, \mathcal{B}_2)$. If $\mathcal{B}$ is a condition implication structure representing a normative system S, where $\mathcal{B}_1$ is the set of descriptive grounds and $\mathcal{B}_2$ is the set of normative consequences, then $Joining_{\mathcal{B}}(\mathcal{B}_1, \mathcal{B}_2)$ can be thought of as the maximal set of norms in $\mathcal{B}$.

A special kind of joining is connections.

Definition 5. *A* connection *from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ is a pair $\langle b_1, b_2 \rangle$ such that following four requirements are satisfied:*

(i) $\langle b_1, b_2 \rangle$ is a joining from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$.

(ii) There is $a_1 \in B_1 \backslash B_2$ and $a_2 \in B_2 \backslash B_1$ such that $a_1 R b_1$ and $b_2 R a_2$.

(iii) If $a_1 \in B_1$ and $b_1 R a_1 R b_2$ then $a_1 R b_1$.

(iv) If $a_2 \in B_2$ and $b_1 R a_2 R b_2$ then $b_2 R a_2$.

Requirements (iii)-(iv) are called the proximity principles. Intuitively, if $\langle b_1, b_2 \rangle$ is a connection, then there is no element in B_1 or B_2 which is strictly between b_1 and b_2. The set of connections from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ will be denoted $Conn_{\mathcal{B}}(\mathcal{B}_1, \mathcal{B}_2)$.

3.2 Two Theorems on Connections

The notion of connection is of particular interest in the study of normative systems. Suppose that the Boolean quasi-ordering $\mathcal{B}$ is the representation of the system $\mathcal{S}$. If $\mathcal{B}_1$ is the set of descriptive conditions while the domain of $\mathcal{B}_2$ consists of the normative conditions, the set of connections from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ can be thought of as representing the specific normative content of $\mathcal{B}$, and modifications of the system implies a change of the sets of connections. The importance of the set of connections can be elucidated using the following notion.

Definition 6. *Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are fragments of $\mathcal{B} = \langle B, \wedge, ', R \rangle$. Then $\trianglelefteq$ is the binary relation on $Joining_{\mathcal{B}}(\mathcal{B}_1, \mathcal{B}_2)$ such that*

$$\langle a_1, a_2 \rangle \trianglelefteq \langle b_1, b_2 \rangle \textit{ if and only if } b_1 R a_1 \textit{ and } a_2 R b_2.$$

The reading of $\langle a_1, a_2 \rangle \trianglelefteq \langle b_1, b_2 \rangle$ is that $\langle a_1, a_2 \rangle$ is at least as narrow as $\langle b_1, b_2 \rangle$. We say that $\trianglelefteq$ is the narrowness relation in $\mathcal{B}$. $\trianglelefteq$ is a quasi-ordering, i.e. transitive and reflexive. Its converse $\trianglerighteq$ is read "at least as wide as".

Let $\triangleleft$ denote the strict part of $\trianglelefteq$. Then the following holds:

$$\langle a_1, a_2 \rangle \triangleleft \langle b_1, b_2 \rangle \text{ if an only if } (b_1 S a_1 \text{ and } a_2 R b_2) \text{ or } (b_1 R a_1 \text{ and } a_2 S b_2).$$

$\langle a_1, a_2 \rangle$ is a *minimal element* in $Joining_{\mathcal{B}}(\mathcal{B}_1, \mathcal{B}_2)$ if and only if there is no $\langle x_1, x_2 \rangle$ in $Joining_{\mathcal{B}}(\mathcal{B}_1, \mathcal{B}_2)$ such that $\langle x_1, x_2 \rangle \triangleleft \langle a_1, a_2 \rangle$.

Theorem 7. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are disjunct fragments of $\mathcal{B}$, then $Conn_{\mathcal{B}}(\mathcal{B}_1, \mathcal{B}_2)$ is the set of minimal joinings from $\mathcal{B}_1$ to $\mathcal{B}_2$ with respect to $\trianglelefteq$.*

The importance of the concept of connections (and thus of minimal joinings) comes from the following observation. Given some general conditions, $\mathcal{B}$ is uniquely determined, by the fragments $\mathcal{B}_1$ and $\mathcal{B}_2$ and the set of connections from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$. Thus if $\mathcal{B}$ is a normative system and $\mathcal{B}_1$ and $\mathcal{B}_2$ are the fragments of grounds and consequences respectively, then the normative content of $\mathcal{B}$ is given by the set of connections from $\mathcal{B}_1$ to $\mathcal{B}_2$, i.e. the set of minimal norms in $\mathcal{B}$.

More exactly, the observation made is expressed by the following theorem. (For the definition of "generated by", see section 5.)

Theorem 8. *Let $\mathcal{B}_1, \mathcal{B}_2$ be fragments of two finite* Bqo*'s $\mathcal{B}^{(1)}, \mathcal{B}^{(2)}$ where,*

1. $B_1 \cap B_2 = \{\bot, \top\}$,
2. $\mathcal{B}_1^{red}$, $\mathcal{B}_2^{red}$ *are generated by sets X_1, X_2, respectively,*
3. $\mathcal{B}^{(1)\,red} = \mathcal{B}^{(2)\,red}$ *where $\mathcal{B}^{(1)\,red}$ is generated by $X_1 \cup X_2$,*
4. $R_1 = R^{(1)}/B_1 = R^{(2)}/B_1$ *and* $R_2 = R^{(1)}/B_2 = R^{(2)}/B_2$.

Then, if $Conn_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2) = Conn_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2)$, it follows that $\mathcal{B}^{(1)} = \mathcal{B}^{(2)}$.

3.3 Comparisons Between Sets of Connections

In the previous section we introduced the relation $\trianglelefteq$, "at least as narrow as", defined for joinings. Using this relation we can now define a binary relation *at least as tight as* between sets of connections in the following way.

Definition 9. *Let $\mathcal{B}_1, \mathcal{B}_2$ be fragments of two finite* Bqo*'s $\mathcal{B}^{(1)} = \langle B, \wedge, ', R^{(1)}\rangle$ and $\mathcal{B}^{(2)} = \langle B, \wedge, ', R^{(2)}\rangle$, and let $\trianglelefteq^{(2)}$ be the narrowness relation in $\mathcal{B}^{(2)}$. Then $Conn_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2)$ is* at least as tight as *$Conn_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2)$ if for any $\langle a_1, a_2\rangle \in Conn_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2)$ there is $\langle b_1, b_2\rangle \in Conn_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2)$ such that $\langle b_1, b_2\rangle \trianglelefteq^{(2)} \langle a_1, a_2\rangle$.*

The converse is *at least as loose as*.

Lemma 10. *Let $\mathcal{B}_1 = \langle B_1, \wedge, ', R_1\rangle, \mathcal{B}_2 = \langle B_2, \wedge, ', R_2\rangle$ be fragments of two finite* Bqo*'s $\mathcal{B}^{(1)} = \langle B, \wedge, ', R^{(1)}\rangle$ and $\mathcal{B}^{(2)} = \langle B, \wedge, ', R^{(2)}\rangle$. Then $Conn_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2)$ is equally tight as $Conn_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2)$ if and only if $Conn_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2) = Conn_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2)$.*

A simple example concerning four different *Bqo*'s $\mathcal{B}^{(1)} - \mathcal{B}^{(4)}$ is as follows:

$$Conn_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2) = \{\langle a_1 \wedge b_1, a_2\rangle\}. \quad Conn_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2) = \{\langle a_1, a_2\rangle\}.$$
$$Conn_{\mathcal{B}^{(3)}}(\mathcal{B}_1, \mathcal{B}_2) = \{\langle b_1, a_2\rangle\}. \quad Conn_{\mathcal{B}^{(4)}}(\mathcal{B}_1, \mathcal{B}_2) = \{\langle a_1 \vee b_1, a_2\rangle\}.$$

$Conn_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2)$ is loosest while $Conn_{\mathcal{B}^{(4)}}(\mathcal{B}_1, \mathcal{B}_2)$ is tightest. System $\mathcal{B}^{(1)}$ is most restrictive, and $\mathcal{B}^{(4)}$ least restrictive, as regards grounds. It holds generally that if $Conn_{\mathcal{B}^{(i)}}(\mathcal{B}_1, \mathcal{B}_2)$ is tighter than $Conn_{\mathcal{B}^{(j)}}(\mathcal{B}_1, \mathcal{B}_2)$, then $\mathcal{B}^{(i)}$ is less restrictive than $\mathcal{B}^{(j)}$ as regards grounds in joinings, or more encompassing as regards consequences in joinings, or both.

Lemma 11. *Suppose that $\mathcal{B}_1 = \langle B_1, \wedge, ', R_1\rangle$, $\mathcal{B}_2 = \langle B_2, \wedge, ', R_2\rangle$ are fragments of two finite* Bqo*'s $\mathcal{B}^{(1)} = \langle B, \wedge, ', R^{(1)}\rangle$ and $\mathcal{B}^{(2)} = \langle B, \wedge, ', R^{(2)}\rangle$. Then, $Joining_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2)$ is a subset of $Joining_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2)$ iff $Conn_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2)$ is at least as loose as $Conn_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2)$.*

Furthermore, as an immediate corollary of lemmas 10 and 11 we get:

Corollary 12. *Suppose that $\mathcal{B}_1 = \langle B_1, \wedge, ', R_1\rangle$, $\mathcal{B}_2 = \langle B_2, \wedge, ', R_2\rangle$ are fragments of two finite* Bqo*'s $\mathcal{B}^{(1)} = \langle B, \wedge, ', R^{(1)}\rangle$ and $\mathcal{B}^{(2)} = \langle B, \wedge, ', R^{(2)}\rangle$. Then,*

$$Joining_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2) = Joining_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2) \text{ iff } Conn_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2) = Conn_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2).$$

Finally, we make the following observation concerning how a set of connections from one fragment to another is related to a set of connections from part of the first fragment to part of the second.

Lemma 13. *Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$ be fragments of a finite* Bqo *$\mathcal{B}$ such that $\mathcal{B}_3 \subseteq \mathcal{B}_1$ and $\mathcal{B}_4 \subseteq \mathcal{B}_2$. Then,*

1. *$Conn_{\mathcal{B}}(\mathcal{B}_1, \mathcal{B}_2)$ is at least as tight as $Conn_{\mathcal{B}}(\mathcal{B}_3, \mathcal{B}_4)$,*
2. *$Conn_{\mathcal{B}}(\mathcal{B}_1, \mathcal{B}_2) \cap (B_3 \times B_4) \subseteq Conn_{\mathcal{B}}(\mathcal{B}_3, \mathcal{B}_4)$.*

3.4 Connections Satisfying a Constraint

As appears from the foregoing, a *Bqo* $\mathcal{B}$ only represents the "positive" implicative statements of a normative system expressed by the relation R. A normative system as conceived of in a wider perspective, however, may have as a feature the existence of special constraints on R. Two kinds of constraints, in particular, will be mentioned here.

Firstly, a constraint on the relation $R^{(1)}$ of a normative system $\mathcal{B}^{(1)}$ (for example, by fundamental principles of admissibility) may refer to two standards of comparison $\mathcal{B}^{(2)}, \mathcal{B}^{(3)}$ and require that $Conn_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2)$ be at least as tight as $Conn_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2)$ and at least as loose as $Conn_{\mathcal{B}^{(3)}}(\mathcal{B}_1, \mathcal{B}_2)$. Due to the lemma 11, such a constraint means that it be required that $Joining_{\mathcal{B}^{(2)}}(\mathcal{B}_1, \mathcal{B}_2) \subseteq Joining_{\mathcal{B}^{(1)}}(\mathcal{B}_1, \mathcal{B}_2) \subseteq Joining_{\mathcal{B}^{(3)}}(\mathcal{B}_1, \mathcal{B}_2)$. It can be the case that all $\mathcal{B}^{(i)}$ not fulfilling the requirement just mentioned are regarded as inadmissible.

Secondly, if $\mathcal{B} = \langle B, \wedge, ', R\rangle$ is any *Bqo* purported to represent the normative system, a special constraint for the admissibility of $\mathcal{B}$ can be introduced by a relation T on B. T is such that if aTb, then not: a implies b. Thus, if the system is consistent, it holds that $T \subseteq \overline{R}$, where $\overline{R}$ is the complement of R. The relation T is a general constraint on the normative system and a requirement on any *Bqo* $\mathcal{B} = \langle B, \wedge, ', R\rangle$ that purports to represent the system. For example, the requirement "Theft does not imply exposure to capital punishment", can be represented by aTb. We note that aTb should be distinguished from aRb'.

4 Conclusion

In the paper we have argued that connections are essential as a kind of "minimal" norms which determine a normative system and are essential for a comparison between different systems. These results go well together with the integration of normative positions within the theory of normative systems, as envisaged in [9]. It remains to develop, however, how the present results on connections influence another aspect of our theory. In [7], a normative system was represented as a class of *Bqo*'s $\mathcal{B}^{(1)}, ..., \mathcal{B}^{(n)}$. One member of the class representing the normative system is the core structure, representing the uncontroversial and settled implicative contents of the system. The other members of the class are conceived of as "amplifications" of the core. These amplifications represent different developments of the law, all of which conform with the core but which differ among themselves on issues which are not settled by the uncontroversial implicative contents. The results of the present paper are applicable to each member $\mathcal{B}^{(i)}$ of the class and can be used for comparisons between members.

5 Appendix

Proof of theorem 7. (I) Suppose that $\langle b_1, b_2\rangle \in Conn(B_1, B_2)$ and $\langle b_1, b_2\rangle$ is not a minimal element with respect to $\trianglelefteq$. Then there is $\langle a_1, a_2\rangle \in Joining(B_1, B_2)$ such that $\langle a_1, a_2\rangle \lhd \langle b_1, b_2\rangle$. From this follows that (1) b_1Sa_1 and a_2Rb_2 or (2) b_1Ra_1 and a_2Sb_2. According to definition 5, (1) implies a_1Rb_1, which is a contradiction. Moreover, (2) implies b_2Ra_2, which is a contradiction.
(II) Suppose that $\langle b_1, b_2\rangle$ is a minimal element in $Joining(B_1, B_2)$ with respect to $\trianglelefteq$. Suppose (iii) in definition 5 is violated. Then there is $a_1 \in B_1$ such that $b_1Sa_1Rb_2$. Hence, $\langle a_1, b_2\rangle \lhd \langle b_1, b_2\rangle$ which implies that $\langle b_1, b_2\rangle$ is not a minimal element in $Joining(B_1, B_2)$. In an analogous way it is proved that if (iv) is violated then we get a contradiction.□

Proof of theorem 8. Before the theorem is proved, we state a definition and a proposition in the theory of Boolean algebras, as well as a lemma in *Bqo* theory.

Definition of a subalgebra generated by a set X. Let $\mathbf{B} = \langle B, \wedge, ' \rangle$ be a Boolean algebra and $X \subseteq B$. Then $\langle \lceil X \rceil, \wedge, ' \rangle$, where

$$\lceil X \rceil = \cap\{A \subseteq B \mid \langle A, \wedge, ' \rangle \text{ is a subalgebra of } \mathbf{B} \ \& \ X \subseteq A\},$$

is the *subalgebra generated by* X in $\mathbf{B}$. (See, for example, [4], p. 51.)

Proposition. Suppose that $\mathbf{B} = \langle B, \wedge, ' \rangle$ is a Boolean algebra generated by $X \cup Y$ and that $\lceil X \rceil$ and $\lceil Y \rceil$ are disjunct. If $a \in B$ then there are $x_1, ...x_\mu \in X$ and $y_1, ...y_\mu \in Y$ such that $a = (x_1 \wedge y_1) \vee ... \vee (x_\mu \wedge y_\mu)$, and (by duality) there are $x_1, ...x_\mu \in X$ and $y_1, ...y_\mu \in Y$ such that $a = (x_1 \vee y_1) \wedge ... \wedge (x_\nu \vee y_\nu)$.

Lemma (*). Suppose that $\mathcal{B} = \langle B, \wedge, ', R \rangle$ is a *Bqo*. Then,

$$(a_1 \wedge c_1) \vee ... \vee (a_\mu \wedge c_\mu) \ R \ (b_1 \vee d_1) \wedge ... \wedge (b_\nu \vee d_\nu)$$

iff for all $i = 1, ..., \mu$, and all $j = 1, ..., \nu$, $(a_i \wedge b'_j) R (c'_i \vee d_j)$.

The proof of this lemma is obvious from the principles holding according to the definition of a *Bqo* (definition 1).

Given these preliminaries, *theorem* 8 *is proved as follows*. Suppose that $\alpha \, R^{(1)} \, \beta$. Then there are $a_1, ..., a_\mu, b_1, ..., b_\nu \in X$ and $c_1, ..., c_\mu, d_1, ..., d_\nu \in Y$ such that $\alpha = (a_1 \wedge c_1) \vee ... \vee (a_\mu \wedge c_\mu)$, $\beta = (b_1 \vee d_1) \wedge ... \wedge (b_\nu \vee d_\nu)$.

From lemma (*) it follows that $\alpha \, R^{(1)} \, \beta$ iff, for all $i \in \{1, ..., \mu\}$, and all $j \in \{1, ..., \nu\}$, $(a_i \wedge b'_j) \, R^{(1)} \, (c'_i \vee d_j)$, where $(a_i \wedge b'_j) \in B_1$ and $(c'_i \vee d_j) \in B_2$. By corollary 12 it follows from the assumption of the theorem that $\left\langle a_i \wedge b'_j, c'_i \vee d_j \right\rangle \in R^{(2)}$. Consequently, according to lemma (*), $\alpha R^{(2)} \beta$. Thus $R_1 \subseteq R_2$. By analogous argument $R_2 \subseteq R_1$. Hence $\mathcal{B}_1 = \mathcal{B}_2$.□

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